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ARPA Order No. 529

ON THE KINETIC THEORY OF DIFFUSION OF A PLASMA COLUMN ACROSS A MAGNETIC FIELD

by

Michael J. Haggerty

Research Report No. PIBMRI-1289-65
Contract No. NONR 839(38)

Sponsored by

Advanced Research Projects Agency

Monitored by

Office of Naval Research
Washington, D. C.
August 10, 1965

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POLYTECHNIC INSTITUTE OF BROOKLYN

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DEPARTMENT OF PHYSICS

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Acknowledgments

The formal solution given in Section 3 was first derived (in a slightly different form) in 1962, as part of a doctoral program at the University of British Columbia, under the active supervision of Dr. L. G. de Sobrino. At that time, I was financially supported by a National Research Council of Canada Studentship.

More recently, I am indebted to Dr. T.-Y. Wu and Dr. D. D. H. Yee for many stimulating conversations. The research was supported by the Office of Naval Research, Washington, D. C. under Contract No. NONR 839(38); ARPA Order No. 529.

Summary

The kinetic theory of Prigogine and Balescu, which was previously developed for a homogeneous plasma in a magnetic field, is now applied to the problem of diffusion of a plasma column across a magnetic field. The problem of deriving a kinetic equation is reduced to that of solving a two dimensional Fredholm-type integral equation with a complicated kernel. No restriction is made concerning the amplitude or length scale of the inhomogeneity across the magnetic field. The results are preliminary in nature, and suggestions are made for further refinements. Some comparisons with previously existing results are given.

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SECTION 1

INTRODUCTION

The phenomenon of diffusion of a plasma across a magnetic field is of great importance in the problem of plasma confinement. There have been many theoretical papers on the subject; see for example the review articles by Golant[1] and Boeschoten[2]. So far, however, there has been very little work done on the kinetic theory of inhomogeneous nonequilibrium plasmas in magnetic fields.

We report here some preliminary results on the diffusion of a plasma column across a uniform magnetic field. It is believed that no equivalent results have yet been published. The problem of deriving an equation for the one particle distribution function, including a proper treatment of screening effects, has been reduced to that of solving an integral equation of the Fredholm type, with a complicated kernel. The method used is an extension of that of Prigogine, Balescu, and coworkers.[3],[4] It was applied to the homogeneous problem in preceding works [5], [6], [7], to be referred to as papers I, II, and III respectively, We rely heavily on the notation summarized in the Appendix of paper I (although some superscripts $\perp$ and z are suppressed, and $\underline{x}_j^{g\perp}$ is replaced by $\underline{Q}_j$). Some familiarity with paper I or with Balescu's book[4] is necessary for a detailed examination of this report; the emphasis is on the differences between the present calculation and that of paper I.

Although the plasma is assumed to be homogeneous along the magnetic field, an arbitrary inhomogeneity is allowed in directions perpendicular to the field. The infinite order perturbation theory is tractable for this case because the guiding centers are constants of the unperturbed motion. We have, however, restricted ourselves here to the calculation of local effects in a single species plasma. It is an important problem to refine this model, and to properly take into account internal macroscopic electric fields.

More details concerning the physical model are given in Section 2. The diagram perturbation theory is summarized in Sections 3, 4, and 5. The main results are given in Sections 6 and 7. They are compared with existing kinetic equations in Section 8.

SECTION 2

PHYSICAL MODEL

The theory presented here is incomplete in the sense that a very primitive physical model is used.

We consider a single-species plasma in a region of volume Ω containing a uniform magnetic field $\underline{B}$. The length of the region in the direction of $\underline{B}$ is Λ, and the cross section is $\Sigma = \Omega/\Lambda$. We will assume that the plasma is homogeneous in the direction of $\underline{B}$. In addition, the limit

$$N \to \infty, \ \Lambda \to \infty, \ N/\Lambda \text{ constant}, \quad \Sigma \text{ arbitrary} \tag{2.1}$$

(cf. eq. (I. 2.1)) will be taken. Thus end effects (for example, "short circuiting") are not considered. However, an arbitrary inhomogeneity in directions perpendicular to $\underline{B}$ will be allowed. There will be no restrictions on the amplitude or length scale of the inhomogeneities, although these scales may effect the plausibility of some of the other assumptions.

A neutralizing background charge of unspecified density variation across $\underline{B}$ is assumed to be present. This is an ill-defined artifice introduced solely for the purpose of restricting ourselves to the calculation of effects due to relatively <u>local</u> interactions among the plasma particles. The long range interactions and boundary effects which the perturbation expansion of paper I cannot take into account are assumed to effectively cancel each other: we again make assumption (I. 3. 26). We use the Hamiltonian (I. 2. 2) with W set equal to zero.

In the homogeneous case discussed in paper I, the generalization to a more realistic multi-component plasma would involve only the inclusion of additional indices and summation signs in appropriate places. In the present case, however, such a generalization may be a less trivial problem. It would involve a careful treatment of the transverse electric fields* arising from the charge separation due to different diffusion rates of the

*Note that for the noninteracting case, the orbits in crossed <u>external</u> electric and magnetic fields are very simple; see for example Reference (8), Section 10.1.

various plasma components. This should lead to the development of a theory of ambipolar diffusion. While such a treatment is important, it is beyond the scope of the work reported here.

We will also assume that the one particle distribution function is initially "gyrotropic"; that is, isotropic in directions perpendicular to $\underline{B}$. This may restrict the types of diffusion that the theory can predict. However, the theorem that the initial condition is preserved in time is much stronger than the corresponding result for the homogeneous plasma. In that case, once it became apparent that the higher order initial correlations would not appear in the final results, then there was no way in which a preferred direction perpendicular to $\underline{B}$ could appear. In the present case, we always have the preferred directions corresponding to the spatial density gradients.

SECTION 3

FORMAL SOLUTION OF THE LIOUVILLE EQUATION

Our starting point is the same as in Section I. 3, except that we do not introduce periodic boundary conditions with respect to the $\{\underline{x}_j^{g\perp}\}$ (hereafter called $\{\underline{\Omega}_j\}$). The fundamental reason why the treatment of arbitrary inhomogeneities perpendicular to $\underline{B}$ is tractable has already been noted in the paragraph following eq. (I. 3. 13). The $\{\underline{\Omega}_j\}$ are constants of the unperturbed motion, and hence can be treated in the same way as the velocity cylindrical components $\{a_j, p_j^z\}$. (From now on, the superscript z will be suppressed.) We had no reason to assume any sort of homogeneity in velocity space (except through the condition of stability); hence none is needed with respect to $\{\underline{\Omega}_j\}$.

Thus eqs. (I. 3. 14) and (I. 3. 18) are replaced by

$$f_N(t) \equiv f_N(\{z_j\}, \{\underline{\Omega}_j\}, \{a_j\}, \{\theta_j\}, \{p_j\}, t)$$

$$= \sum_{\{k\}} \sum_{\{n\}} \rho'_{\{k\}\{n\}}(t) \exp\left[i \sum_j (k_j z_j - n_j \theta_j)\right], \qquad (3.1)$$

$$\rho'_{\{k^{(o)}\}\{n^{(o)}\}}(t) \equiv \rho'_{\{k^{(o)}\}\{n^{(o)}\}}(\{\underline{\Omega}_j\}, \{a_j\} \{p_j\}, t)$$

$$= -\frac{1}{2\pi i} \oint_C d\zeta \exp(-i\zeta t) \sum_{q=o}^{\infty} \{ (-e^2)^q$$

$$[(\sum_j k_j^{(o)} p_j/m) + \omega(\sum_i n_j^{(o)}) - \zeta]^{-1}$$

$$\sum_{\{k^{(1)}\}} \sum_{\{n^{(1)}\}} \sum_{h<u} \langle \{k_j^{(o)}\} \{n_j^{(o)}\} | \delta L_{hu} | \{k_j^{(1)}\} \{n_j^{(1)}\} \rangle$$

$$[(\sum_j k_j^{(1)} p_j/m) + \omega(\sum_j n_j^{(1)}) - \zeta]^{-1}.$$

$$\ldots\ldots \sum_{\{k^{(q)}\}} \sum_{\{n^{(q)}\}} \sum_{i<m}$$

$$< \{k_j^{(q-1)}\} \{n_j^{(q-1)}\} |\delta L_{im}| \{k_j^{(q)}\}\{n_j^{(q)}\} >$$

$$[(\sum_j k_j^{(q)} p_j/m) + \omega(\sum_j n_j^{(q)}) - \zeta]^{-1} \rho'_{\{k^{(q)}\}\{n^{(q)}\}}{}^{(0)} \quad . \tag{3.2}$$

Here, the k's are restricted to values of the form $(2\pi/\Lambda)$(integer), and the n's are integers.

The matrix elements are defined as follows:

$$<\{k_j\}\{n_j\} \; |\delta L_{hu}|\{k_j'\}\{n_j'\} \; >$$

$$= \Lambda^{-N}(2\pi)^{-N} \int d^N z \; d^N \theta$$

$$\exp[-i\sum_j (k_j z_j - n_j \theta_j)] \; \delta L_{hu} \; \exp[i \sum_j (k_j' z_j - n_j' \theta_j)].$$

They are operators with respect to $\{Q_j\}$, $\{a_j\}$, and $\{p_j\}$. An explicit expression analogous to eq. (I. 3. 25) can be calculated in the same way:

$$<\{k_j\} \{n_j\} |\delta L_{hu}|\{k_j'\} \; \{n_j'\} >$$

$$= [\prod_{j(\neq h, u)} \delta^{Kr}(k_j - k_j') \, \delta^{Kr}(n_j - n_j')]$$

$$\delta^{Kr}(k_h + k_u - k_h' - k_u') \int_0^\infty \ell d\ell \int_0^{2\pi} d\beta$$

$$[-\pi m\omega\Lambda(\ell^2 + [k_h - k_h']^2 + \varkappa^2)]^{-1}$$

$$\exp[i(n_h + n_u - n_h' - n_u')\beta] \Big\{ J_{n_h - n_h'}(\ell a_h)$$

$$J_{-(n_u - n_u')}(\ell a_u)\,[m\omega(k_h - k_h')\,(\frac{\partial}{\partial p_h} - \frac{\partial}{\partial p_u})$$

$$-(\underline{\ell} \times \frac{\partial}{\partial \underline{Q}_{hu}})^z + \frac{n_h}{a_h}\frac{\partial}{\partial a_h} + \frac{n_u}{a_u}\frac{\partial}{\partial a_u}]$$

$$-J_{-(n_u - n_u')}(\ell a_u)\,\frac{n_h'}{a_h}\frac{\partial}{\partial a_h} J_{n_h - n_h'}(\ell a_h)$$

$$-J_{n_h - n_h'}(\ell a_h)\,\frac{n_u'}{a_u}\frac{\partial}{\partial a_u} J_{-(n_u - n_u')}(\ell a_u)\Big\}$$

$$\exp[i\,\underline{\ell}\cdot\underline{Q}_{hu}], \qquad (3.3)$$

where $\underline{Q}_{hu} = \underline{Q}_h - \underline{Q}_u$, and $\underline{\ell}$ is the vector defined by the polar coordinates ℓ, β ($\underline{\ell}\cdot\underline{B} = 0$).

It is possible to perform the integration over β. Although the result will not be used, we write it for completeness:

$$< \{k_j\}\ \{n_j\}\ |\delta L_{hu}|\{k_j'\}\ \{n_j'\} >$$

$$= [\prod_{j(\neq h,u)} \delta^{Kr}(k_j - k_j')\,\delta^{Kr}(n_j - n_j')]$$

$$\delta^{Kr}(k_h + k_u - k_h' - k_u')\, 2\pi \int_0^\infty \ell\, d\ell$$

$$[-\pi m\omega\Lambda(\ell^2 + [k_h - k_h']^2 + \varkappa^2)]^{-1}$$

$$\Big\{ [m\omega(k_h - k_h')(\frac{\partial}{\partial p_h} - \frac{\partial}{\partial p_u}) - \frac{n_h'}{a_h}\frac{\partial}{\partial a_h}$$

$$- \frac{n_u'}{a_u}\frac{\partial}{\partial a_u} + \frac{n_h + n_u - n_h' - n_u'}{Q_{hu}}\frac{\partial}{\partial Q_{hu}}$$

$$+ \frac{i\ell}{Q_{hu}} \frac{J'_{n_h + n_u - n_h' - n_u'}(\ell Q_{hu})}{J_{n_h + n_u - n_h' - n_u'}(\ell Q_{hu})} \frac{\partial}{\partial \phi_{hu}}]$$

$$\cdot [J_{n_h - n_h'}(\ell a_h)\, J_{-(n_u - n_u')}(\ell a_u)] + [J_{n_h - n_h'}(\ell a_h)$$

$$J_{-(n_u - n_u')}(\ell a_u)][\frac{n_h}{a_h}\frac{\partial}{\partial a_h} + \frac{n_u}{a_u}\frac{\partial}{\partial a_u}]\Big\}$$

$$J_{n_h + n_u - n_h' - n_u'}(\ell Q_{hu}) \exp[i(n_h + n_u - n_h' - n_u')(\phi_{hu} + \frac{\pi}{2})], \qquad (3.4)$$

where $\phi_{hu} = \arctan(Q_{hu}^y / Q_{hu}^x)$ (cf. eq. (I. 3. 8) ff).

We also note that, according to Barabanenkov[9], eqs. (6) and (7), the integration over ℓ is simple if $Q_{hu} > a_h + a_u$. One should however check the range of validity of those formulae before applying them.

SECTION 4

INITIAL CONDITIONS

For the physical model described in Section 2, the arguments in Sections I. 4 and I. 5 go through almost without change. All that is necessary is to replace Ω or $\Omega/8\pi^3$ by Λ or $\Lambda/2\pi$ in various places, and to include integrations over $\{ \underline{Q}_j \}$ with the velocity integrals.

The reduced distribution functions can be defined as before:

$$f_{r,s}(z_1, \ldots, z_r; \{\underline{Q}_\sigma\}, \{a_\sigma\}, \{\theta_\sigma\}, \{p_\sigma\}; t)$$

$$= \frac{N!}{(N-r)!} \int'_{(1,\ldots,r)} d^{N-r}z \int'_{\{\sigma\}} d^{N-s}\underline{Q}\, d^{N-s}a\, d^{N-s}\theta\, d^{N-s}p$$

$$[\prod_{h=s+1}^{N} m^2\omega^2 a_h]\, f_N(\{z_j\}, \{\underline{Q}_j\}, \{a_j\}, \{\theta_j\}, \{p_j\}, t); \tag{4.1}$$

$$\varphi_s = f_{o,s}, \varphi = \varphi_1 = f_{o,1}; \; 0 \le r \le s \le N;$$

with the normalization condition

$$\int d^s\underline{Q}\; d^sa\; d^s\theta\; d^sp\, (\prod_\sigma m^2\omega^2 a_\sigma)\, \varphi_s(\{\sigma\}, t) = 1. \tag{4.2}$$

They can be expressed in terms of the ρ''s:

$$f_{r,s} = \Lambda^N (N/\Lambda)^r \sum_{k_1 \ldots k_r} \sum_{\{n_\sigma\}} \mathscr{Y}_{N-s}$$

$$\rho'^{\,1\ldots r\;\{\sigma\}}_{\,k_1\ldots k_r\;\{n\}} \exp(-i \sum_\sigma n_\sigma \theta_\sigma)$$

$$\exp(i[k_1 z_1 + \ldots + k_r z_r]). \tag{4.3}$$

which is analogous to eq. (I. 4. 7). Here,

$$\mathcal{Y}_{N-s} f \equiv \int_{\{\sigma\}}' d^{N-s} \underline{Q}\ d^{N-s} a\ d^{N-s} p\ \left[\prod_{h=s+1}^{N} 2\pi m^2 \omega^2 a_h\right] f \ . \qquad (4.4)$$

The Ursell correlation functions $U_s(\{\sigma\})$ are defined by eqs. (I. 4.11), with Ω replaced by Λ.

Our object is to obtain equations for the one particle distribution function φ. Initial conditions analogous to those of Section I. 5 are imposed:

$$f_{1,s}(0) = f_{r,r}(0) \prod_{\tau=r+1}^{s} \varphi(\underline{Q}_\tau, a_\tau, \theta_\tau, p_\tau; 0), \qquad (4.5)$$

$$\varphi(\underline{Q}_\tau, a_\tau, \theta_\tau, p_\tau; 0) = \varphi(\underline{Q}_\tau, a_\tau, p_\tau; 0), \qquad (4.6)$$

$U_s(\{\sigma\}, 0)$ is finite in the limit (2.1), and

$$\lim_{\Lambda\to\infty} U_s(\ldots,\alpha,\ldots,\beta,\ldots;0) = 0 \text{ for } |z_\alpha - z_\beta| >> \varkappa_{corr}^{-1}, \qquad (4.7)$$

$$f_N(\{z_j\}, \ldots, 0) = f_N(\{z_j + Z\}, \ldots, 0). \qquad (4.8)$$

It is assumed that there are no initial correlations among the guiding centers or velocities which are independent of the distances $\{|z_i - z_j|\}$, in the limit $\Lambda \to \infty$.

The condition (4.6) is stronger than for the homogeneous case, because of the preferred directions corresponding to density gradients across $\underline{B}$. However, a strong theorem can be established; namely, that the condition is preserved in time in the modified weak coupling approximation.

We can then easily establish the analogue of the result (I. 5.22): If

$$\rho^{1\ldots r}_{k\ldots q}\begin{Bmatrix}\sigma\\ n\end{Bmatrix} \equiv \Lambda^{N}(\Lambda/2\pi)^{r-w}\,\rho'^{1\ldots r}_{k\ldots q}\begin{Bmatrix}\sigma\\ n\end{Bmatrix},$$

$$k+\ldots+q = 0,\ k \neq 0, \ldots, q \neq 0, \tag{4.9}$$

then

$$(2\pi/\Lambda)^{w'}\ \mathcal{Y}_{N-s}\rho^{1\ldots r}_{k\ldots q}\begin{Bmatrix}\sigma\\ n\end{Bmatrix}(0) \text{ remains finite in the limit (2.1),} \tag{4.10}$$

where w and w′ have the same meaning as before. This follows from expressions such as that analogous to (I.5.13):

$$U_2(1,2;0) = (2\pi/\Lambda)\sum_{k(\neq 0)}\ \sum_{\{n\}}\ \mathcal{Y}_{N-2}$$

$$\rho^{1\ 2\ 1\ 2}_{k\,-k\,n_1\,n_2}(0)\exp[-i\sum_{\sigma} n_\sigma\,\theta_\sigma]\exp[ik(z_1-z_2)]\,. \tag{4.11}$$

SECTION 5

SELECTION OF THE DOMINANT TERMS

The diagram representation of the formal solution (3.1), (3.2), (3.3) is almost identical to that for the homogeneous case. The "semi-connection" property (I. 7.7) remains valid, because although there is an extra term in (3.3) containing $\underline{\ell} \times \partial/\partial \underline{Q}_{hu}$, there is no dependence on $\underline{Q}_h$ or $\underline{Q}_u$ to the left of this factor.

By calculating the Λ dependences of the various diagram contributions to $\mathcal{Y}_{N-s} \rho_{\{k\}}{}^{\{\sigma\}}_{\{n\}}(t)$ (using condition (4.10)), we can establish that condition (4.10) remains valid for $t > 0$.

As for the e^2 dependence of the initial distributions, there is no essential change. In equations such as (I. 7.20), one replaces $\int d^N \underline{x}^g / \Omega^N$ by $\int d^N z / \Lambda^N$, and $(4\pi^2/\Sigma) \sum_{\underline{\ell}^\perp}$ by $\int d\underline{\ell}^\perp$. (It is then possible to integrate over θ to obtain a simpler result; cf. eq. (3.4).) We therefore assume that

$$\rho'_{\{k\}\ \{n\}}{}^{(0)} \propto (e^2)^{r-w-w'} \rho'_{\{0\}\{0\}}{}^{(0)}, \tag{5.1}$$

which is similar to eq. (I. 7.24).

Let us write the equation for the contribution to $\varphi(\alpha, t)$ corresponding to (I. 8.1) or (I. 8.3):

$$\varphi(\alpha, t) = \ldots - \mathcal{Y}_{N-1} (2\pi i)^{-1} \oint_C d\zeta \exp(-i\zeta t)$$

$$(-\zeta)^{-1} \sum_k \sum_{n_\alpha^{(1)}} \sum_{n_h^{(1)}} (-Ne^2)$$

$$< \{0\}\{0\} \; | \delta L_{h\alpha} | \; k_h = -k,\ k_\alpha = k,\ n_\alpha^{(1)},\ n_h^{(1)},\ \{0\}' >$$

$$(k[p_\alpha - p_h]/m + \omega[n_\alpha^{(1)} + n_h^{(1)}] - \zeta)^{-1} \sum_{n_u^{(2)}} (-Ne^2)$$

$$< k_h = -k, k_\alpha = k,\ n_\alpha^{(1)},\ n_h^{(1)},\ \{0\}' \,|\delta L_{uh}|\, k_u = -k, k_\alpha = k,$$

$$n_\alpha^{(1)},\ n_u^{(2)},\ \{0\}' > (k[p_\alpha - p_u]/m + \omega\,[n_\alpha^{(1)} + n_u^{(2)}] - \zeta)^{-1}$$

$$(-e^2) < k_u = k,\ k_\alpha = k,\ n_\alpha^{(1)},\ n_u^{(2)},\ \{0\}' \,|\delta L_{\alpha u}|\, \{0\}\{0\} >$$

$$(-\zeta)^{-1}\ {}^{o}\{0\}\{0\}^{(0)}, \tag{5.2}$$

where $\{0\}'$ means that all suppressed variables are zero. We replace $n_\alpha^{(1)}$, $n_h^{(1)}$, $n_u^{(2)}$ by n_α, $-n_h$, $-n_u$, and make the following abbreviations in notation:

$$< 0\ |\alpha h|\ > \equiv -Ne^2 i \sum_{n_\alpha} \sum_{n_h}$$

$$\cdot < \{0\}\{0\}\ |\delta L_{h\alpha}|\, k_h = -k,\ k_\alpha = k,\ n_\alpha^{(1)} = n_\alpha,\ n_h^{(1)} = -n_h,\ \{0\}' >$$

(with certain terms that vanish under $\mathcal{Y}_{N-1}$ omitted)

$$\equiv \sum_{n_\alpha} \sum_{n_h} \int_0^\infty \ell_h\, d\ell_h \int_0^{2\pi} d\beta_h \exp(i[n_n - n_\alpha]\,\beta_h)$$

$$\frac{e^2 N \exp(i\,\underline{\ell}_h \cdot \underline{\Omega}_{h\alpha})}{i\pi m\omega\Lambda(\ell_h^2 + k^2 + \varkappa^2)} \left[m\omega k \frac{\partial}{\partial p_\alpha}\right.$$

$$\left. -(\underline{\ell}_h \times \frac{\partial}{\partial \underline{\Omega}_\alpha})^z + \frac{n_\alpha}{a_\alpha} \frac{\partial}{\partial a_\alpha}\right] J_{n_\alpha}(\ell_h a_\alpha)\, J_{n_h}(\ell_h a_h), \tag{5.3}$$

$$< | hu | > \equiv - Ne^2 i \sum_{n_u} < k_h = -k, k_\alpha = k,$$

$$n_\alpha^{(1)} = n_\alpha, \; n_h^{(1)} = -n_h, \{ 0 \}' | \delta L_{uh} | k_u = -k, k_\alpha = k, \; n_\alpha^{(2)} = n_\alpha, \; n_u^{(2)} = -n_u, \{0\}' >$$

(with similar terms omitted)

$$\equiv \sum_{n_u} \int_0^\infty \ell_u \, d\ell_u \int_0^{2\pi} d\beta_u \exp(i[n_u - n_h]\beta_u)$$

$$\frac{e^2 N \exp(i \underline{\ell}_u \cdot \underline{\Omega}_{uh})}{i\pi m\omega\Lambda(\ell_u^2 + k^2 + \varkappa^2)} J_{n_h}(\ell_u a_h) J_{n_u}(\ell_u a_u)$$

$$[m\omega k \frac{\partial}{\partial p_h} - (\underline{\ell}_u \times \frac{\partial}{\partial \underline{\Omega}_h})^z + \frac{n_h}{a_h} \frac{\partial}{\partial a_h}], \tag{5.4}$$

$$< | u\alpha | 0 > \equiv - \frac{\Lambda e^2 i}{2\pi} < k_u = -k, k_\alpha = k,$$

$$n_\alpha^{(2)} = n_\alpha, n_u^{(2)} = - n_u, \; \{0\}' | \delta L_{\alpha u} | \{0\} \; \{0\} >$$

$$\equiv \int_0^\infty \ell \, d\ell \int_0^{2\pi} d\beta \exp(i[n_\alpha - n_u]\beta)$$

$$\frac{e^2 \exp(i\underline{\ell} \cdot \underline{\Omega}_{\alpha u})}{2i\pi^2 m\omega(\ell^2 + k^2 + \varkappa^2)} J_{n_u}(\ell a_u) \, J_{n_\alpha}(\ell a_\alpha)$$

$$\cdot [m\omega k(\frac{\partial}{\partial p_u} - \frac{\partial}{\partial p_\alpha}) - (\underline{\ell} \times \frac{\partial}{\partial \underline{\Omega}_{u\alpha}})^z$$

$$+ \frac{n_u}{a_u} \frac{\partial}{\partial a_u} - \frac{n_\alpha}{a_\alpha} \frac{\partial}{\partial a_\alpha}]. \tag{5.5}$$

Then eq. (5.2) becomes:

$$\varphi(\alpha, t) = \ldots + \mathcal{Y}_{N-1} \frac{i}{2\pi} \oint_C d\zeta \exp(-i\zeta t)$$

$$(i\zeta^2)^{-1} \frac{2\pi}{\Lambda} \sum_k < 0 \, |\alpha h| > (i[G_{\alpha h} - \zeta])^{-1}$$

$$< |hu| > (i[G_{\alpha u} - \zeta])^{-1} < |u\alpha| \, 0 > \rho_{\{0\}\{0\}}(0), \tag{5.6}$$

where $\rho_{\{0\}\{0\}}(0) \equiv \Lambda^N \rho'_{\{0\}\ \{0\}}(0)$ can be replaced by $\prod_j \varphi(j, 0)$ (cf. eq. (I.10.3)).

The arguments from eq. (I.8.8) to eq. (I.8.18) can then be repeated. In the asymptotic limit $t >> t_c$, eq. (5.6) becomes

$$\varphi(\alpha, t) = \ldots + t \int_{-\infty}^{\infty} dk \quad \mathcal{Y}_{N-1} < 0|\alpha h|> \delta_-(G_{\alpha h}) <|hu|> \delta_-(G_{\alpha u}) <|u\alpha|0> \prod_j \varphi(j, 0) \,. \tag{5.7}$$

In selecting the dominant terms, we again make the modified weak coupling approximation. Out of all the diagrams proportional to particular powers of N/Λ and t, we select the ones proportional to the lowest power of e^2. Then the arguments of Section I.9 can be repeated (see, however, the footnote at the end of Section II.2).

The dominant diagrams are semiconnected chains of ring diagrams.

In particular, we have the result analogous to eq. (I. 9. 9):

$$\text{If } k_j^{(\lambda)} = 0, \text{ then } n_j^{(\lambda)} = 0, \text{ for any } j \text{ and } \lambda. \tag{5.8}$$

This implies that the isotropy condition (4. 6) is preserved in time.

SECTION 6

THE SINGULAR INTEGRAL EQUATION

The major mathematical complication arising from the presence of the inhomogeneity across $\underline{B}$ is already apparent from eq. (5.7). The matrix elements contain two labels instead of one, corresponding to the labels to the right and left of the diagram vertex. This feature cannot be rectified by shifting the position of the Bessel functions, as was done for the homogeneous case, because of the presence of different dummy variables $\underline{\ell}_h$, $\underline{\ell}_u$, and $\underline{\ell}$, instead of a single variable $\underline{k}$.[+]

This complication seems to prevent the derivation of a kinetic equation of the same simple form as for the homogeneous case. However, some progress has been made, as will be seen below.

It can be shown that the analogue of eq. (I. 10. 1) is

$$\varphi(\alpha, t) = \ldots + \mathcal{Y}_{N-1}\, t \int_{-\infty}^{\infty} dk < 0|\alpha 1| >$$

$$\cdot \delta\_(G_{\alpha 1}) < |12| > \delta\_(G_{\alpha 2}) < |\alpha 3| \overset{*}{>} \delta\_(G_{32})$$

$$\cdot < |34| \overset{*}{>} \delta\_(G_{42}) < |45| \overset{*}{>} \delta\_(G_{52}) < |26| >$$

$$\cdot \delta\_(G_{56}) < |57| \overset{*}{>} \delta\_(G_{76}) < |68| > \delta\_(G_{78})$$

$$\cdot < |79| \overset{*}{>} \delta\_(G_{98}) < |8\ 10| > \delta\_(G_{9\ 10}) < |10\ 11| >$$

$$\cdot \delta\_(G_{9\ 11}) < |9\ 12| \overset{*}{>} \delta\_(G_{12\ 11}) < |11\ 13| >$$

$$\cdot \delta\_(G_{12\ 13}) < |13\ 12|\ 0 > \prod_j \varphi(j, 0), \tag{6.1}$$

where the asterisks denote the complex conjugate. The arguments leading to eq. (I. . .15) are not affected by the more complicated expressions for the vertices; the cha... of ring diagrams can be factorized as before. In particular the condition (4.5) is preserved in time, at least for $r = 0$.

However, we cannot express the results in a form exactly analogous to eqs. (I.10.18), (I.10.25). Instead, we define a new quantity $F(\alpha, \gamma)$, which is related to the <u>binary correlation function</u> by a Fourier transformation:

$$U_2(\alpha, \gamma) = \int_{-\infty}^{\infty} dk \sum_{n_\alpha} \sum_{n_\gamma} F(\alpha, \gamma) \exp(ik[z_\alpha - z_\gamma]) \exp(-i[n_\alpha \theta_\alpha - n_\gamma \theta_\gamma]). \tag{6.2}$$

(We suppress the parameters k and t from the arguments from now on, since the equations no longer involve the initial time $t = 0$.) Then it can be shown that

$$\frac{\partial \varphi(\alpha)}{\partial t} = \int_{-\infty}^{\infty} dk \int_h < 0 \,|\alpha h|\, > F(\alpha, h), \tag{6.3}$$

$$F(\alpha, h) = q(\alpha, h) + \delta_-(G_{\alpha h}) \int_u < |hu| > \varphi(h)\, F(\alpha, u)$$

$$+ \delta_-(G_{\alpha h}) \int_u <|\alpha u|\overset{*}{>} \varphi(\alpha)\, F^*(u, h), \tag{6.4}$$

where

$$q(\alpha, h) \equiv \delta_-(G_{\alpha h}) < |h\alpha|\ 0 > \varphi(h)\, \varphi(\alpha), \tag{6.5}$$

$$\int_j f \equiv \int_\Sigma dQ_j \int_0^{\infty} da_j \int_{-\infty}^{\infty} dp_j\ 2\pi\, m^2 \omega^2 a_j f. \tag{6.6}$$

Note that if the matrix elements did not depend on the second labels, then one could express $\partial\varphi(\alpha)/\partial t$ in terms of $\int_h F(\alpha, h)$, and derive an integral equation for $\int_h F(\alpha, h)$.

The singular integral equation (6.4) is very complicated, and does not seem readily amenable to the treatment described in Section II.4. By making transformations similar to those in Section 7 below, we could eliminate some of the variables by a sort of "barring" operation, and derive a very lengthy equation for a quantity $\Phi(\alpha, \underline{q}_\gamma)$ which is independent of a_γ, p_γ, and which determines $F(\alpha, \gamma)$ through the following relations:

$$F(\alpha, \gamma) = \int d\underline{q}_\alpha \int d\underline{q}_\gamma \exp[i(\underline{q}_\alpha \cdot \underline{Q}_\alpha - \underline{q}_\gamma \cdot \underline{Q}_\gamma)]\, \hat{F}(\alpha, \gamma), \tag{6.7}$$

and similarly for $q(\alpha, \gamma)$ and $\hat{q}(\alpha, \gamma)$;

$$\hat{F}(\alpha, \gamma) = \hat{q}(\alpha, \gamma) + \Bigg\{ \frac{4\pi e^2 N}{im\omega\Lambda} \underline{\delta}(G_{\alpha\gamma}) \int_0^\infty \frac{q_\varepsilon dq_\varepsilon}{q_\varepsilon^2 + k^2 + \varkappa^2}$$

$$\int_0^{2\pi} d\psi_\varepsilon J_{n_\gamma}(q_\varepsilon a_\gamma) \exp(-in_\gamma \psi_\varepsilon)[m\omega k \frac{\partial}{\partial p_\gamma}$$

$$+ i[\underline{q}_\varepsilon \times \underline{q}_\gamma]^z + \frac{n_\gamma}{a_\gamma} \frac{\partial}{\partial a_\gamma}]\, \hat{\varphi}(\gamma; \underline{k}_\gamma = \underline{q}_\varepsilon - \underline{q}_\gamma))$$

$$\sum_{\nu_\alpha} J_{n_\alpha + \nu_\alpha}(q_\alpha a_\alpha) J_{\nu_\alpha}(q_\alpha a_\alpha) \exp(in_\alpha \psi_\alpha)\, \Phi(\alpha, \underline{q}_\varepsilon) \Bigg\}$$

$$+ \Big\{ \alpha \longleftrightarrow \gamma \Big\}^*, \tag{6.8}$$

where $\hat{\varphi}$ is the Fourier transform of φ:

$$\varphi(\gamma) = \int d\underline{k}_\gamma \exp(i\underline{k}_\gamma \cdot \underline{Q}_\gamma)\, \hat{\varphi}(\gamma), \qquad (6.9)$$

and $\underline{q}_\varepsilon$ is defined by the polar coordinates q_ε, ψ_ε. Such transformations have been used extensively by Rostoker and coworkers, and others; see for example Reference (10).

We do not bother to write out the equation for $\Phi(\alpha, \underline{q}_\gamma)$, because a simpler set of equations can be found in another way. In the next section, we use the summation procedure developed by Résibois [11] in order to derive equations equivalent to (6.3), (6.4), but having a more explicit form.

SECTION 7

THE FREDHOLM-TYPE INTEGRAL EQUATIONS

The Résibois summation procedure[11] has been described very clearly and concisely by Balescu (Reference (4), Appendix 10). Singular integral equations are avoided, and kinetic equations are obtained directly. The method is largely independent of the nature of the matrix elements (vertices). It generalizes immediately to the present physical model, provided that we can consider the integrations term by term in the n's. The n's are held fixed until we actually sum the "primitive diagrams".

The complication mentioned in the last section also appears here, but we are able to proceed much farther. We do not get simple geometric series, and therefore no simple kinetic equation is obtained. However, we can reduce the problem to that of solving a two dimensional integral equation of the Fredholm type but with a complicated kernel. Further progress has not been seriously attempted at the time of writing; it would seem that some approximation may be necessary for the derivation of a kinetic equation in a closed form.

We replace $\delta_-(G_{\alpha\gamma})$ by the equivalent form $[i(G_{\alpha\gamma} - i\epsilon)]^{-1}$, and write, according to the Résibois theorem,

$$F(\alpha, h) = \mathcal{Y}_{N-2} \frac{1}{2\pi} \oint_{C'} d\zeta' \sum_{I,S} \mathcal{V}_{\alpha I}(\zeta') \mathcal{V}^{hS}(i\epsilon - \zeta')$$

$$\cdot < |SI| 0 > \prod_j \varphi(j), \tag{7.1}$$

where

$$\mathcal{V}_{\alpha I}(\zeta') = [i(G_\alpha - \zeta')]^{-1} <|\alpha 2|>^*$$

$$\cdot [i(G_2 - \zeta')]^{-1} < |23| >^* \dots [i(G_I - \zeta')]^{-1},$$

$$\mathcal{V}^{hS}(i\epsilon - \zeta') = [i(-G_h - i\epsilon + \zeta')]^{-1} <|h\ I+1|>$$

$$[i(-G_{I+1} - i\epsilon + \zeta')]^{-1} < |I + 1\; I + 2| >$$

$$\ldots [i(-G_s - i\epsilon + \zeta')]^{-1} .$$

Here, C′ is a path of integration which consists of a straight line from $\zeta' = -\infty + i\epsilon/2$ to $\zeta' = +\infty + i\epsilon/2$, together with the usual semicircle in the lower half plane. (The orientation is opposite to that of path C in eq. (5.2).)

From eq. (7.1), it is a straightforward matter to obtain recurrence relations, and the resulting integral equations:

$$F(\alpha, h) = \frac{1}{2\pi} \oint_{C'} d\zeta' \; \Gamma(\alpha, h), \tag{7.2}$$

$$\Gamma(\alpha, h) = [i(G_\alpha - \zeta')]^{-1} [\Delta(\alpha, h) + \int_u < |\alpha u| >^* \varphi(\alpha) \Gamma(u, h)] , \tag{7.3}$$

$$\Delta(\alpha, h) = [i(-G_h - i\epsilon + \zeta')]^{-1} [\Theta(\alpha, h) + \int_u < |hu| > \varphi(h) \Delta(\alpha, u)] , \tag{7.4}$$

$$\Theta(\alpha, h) = < |h\alpha| 0 > \varphi(h) \varphi(\alpha). \tag{7.5}$$

It is claimed that the set of equations (6.3), (7.2), (7.3), (7.4), (7.5) is simpler than the set (6.3), (6.4), (6.5). Note that if one can solve eq. (7.4), then there is no trouble in solving eq. (7.3), because it has essentially the same kernel.

Let us now investigate briefly the ζ' integration in eq. (7.2). It is relatively easy to perform the integration for the $[i(G_\alpha - \zeta')]^{-1} \Delta(\alpha, h)$ contribution (eq. (7.3)).

The factor $[i(-G_h - i\epsilon + \zeta')]^{-1}$ in eq. (7.4) has no poles in the area enclosed by C! Hence it is very plausible to assume that $\Delta(\alpha, h)$ has no singularities there either; at worst there is some sort of stability condition involved. We can write

$$\frac{1}{2\pi} \oint_{C'} d\zeta' [i(G_\alpha - \zeta')]^{-1} \Delta(\alpha, h) = \Delta(\alpha, h)|_{\zeta' = G_\alpha}$$

$$\equiv \tilde{\Delta}(\alpha, h), \tag{7.6}$$

where

$$\tilde{\Delta}(\alpha, h) = \delta\_(G_{\alpha h})[\Theta(\alpha, h) + \int_u <|hu|> \varphi(h)\, \tilde{\Delta}(\alpha, u)]. \tag{7.7}$$

The other contribution to $\Gamma(\alpha, h)$ is more difficult to calculate. We merely note that the contribution from the semicircular portion of C' may be neglected, and then change notation. Replacing ζ' by $G_x + i\epsilon/2$, we get:

$$F(\alpha, h) = \tilde{\Delta}(\alpha, h) + \int_{-\infty}^{\infty} dG_x\, [\delta\_(G_{\alpha x}) \int_u <|\alpha u|>^* \varphi(\alpha)\, \Gamma(u, h;\, x)] \,, \tag{7.8}$$

$$\Gamma(\alpha, h; x) = \delta\_(G_{\alpha x})\, [\Delta(\alpha, h; x) + \int_u <|\alpha u|>^* \varphi(\alpha)\, \Gamma(u, h;\, x)] \,, \tag{7.9}$$

$$\Delta(\alpha, h; x) = \delta\_(G_{xh})\, [\, \Theta(\alpha, h) + \int_u <|hu|> \varphi(h)\, \Delta(\alpha, h; x)\,] \,, \tag{7.10}$$

$$\tilde{\Delta}(\alpha, h) = \Delta(\alpha, h;\alpha) = \int_{-\infty}^{\infty} dG_x \ \delta(C_{\alpha x})\Delta(\alpha, h;x). \tag{7.11}$$

We now examine the problem of simplifying the integral equations(7. 3) and (7. 4), or (7. 9) and (7. 10). All these equations have essentially the same kernel, and can in principle be solved in the same way. We consider eq. (7. 10) as an example, and abbreviate $\Delta(\alpha, h;x)$ by $\Delta(h)$, since α and G_x appear only as parameters. Also, $\delta_-(G_{xh}) \ \circledcirc \ (\alpha, h)$ will be written as $\Xi(h)$, so that

$$\Delta(h) = \Xi(h) + \delta_-(G_{xh}) \int_u <|hu| > \varphi(h)\, \Delta(u). \tag{7.12}$$

The following expansions are carried out for $\Xi(h)$ as well as for $\Delta(h)$:

$$\Delta(h) = \int dq_h \ \exp(-iq_h \cdot Q_h)\, \hat{\Lambda}(h),$$

(so that $\hat{\Delta}(h) = (2\pi)^{-2} \int dQ_h \ \exp(iq_h \cdot Q_h)\, \Delta(h))$,

$$\varphi(h) = \int dk_h \ \exp(ik_h \cdot Q_h)\, \hat{\varphi}(h);$$

$$\hat{\Delta}(h) = \sum_{\nu_h} J_{n_h + \nu_h}(q_h a_h)\, J_{\nu_h}(q_h a_h) \ \exp(-i n_h \psi_h)\, \hat{\Delta}'(h) \ , \tag{7.13}$$

where $\psi_h = \arctan(q_h^y / q_h^x)$, and $\hat{\Delta}'(h)$ depends on ν_h instead of n_h. Then

$$\hat{\Delta}(h) = \hat{\Xi}(h) + \delta_-(G_{xh}) \frac{4\pi e^2 N}{i m \omega \Lambda} \int_0^\infty \frac{q_u\, dq_u}{q_u^2 + k^2 + \varkappa^2} \int_0^{2\pi} d\psi_u\, J_{n_h}(q_u a_h) \exp(-i n_h \psi_u) \int d\underline{k}_h$$

$$[m\omega k \frac{\partial}{\partial p_h} - i(\underline{q}_u \times \underline{k}_h)^z + \frac{n_h}{a_h} \frac{\partial}{\partial a_h}]\, \delta(\underline{q}_h + \underline{k}_h - \underline{q}_u)\, \hat{\varphi}(h)\, \bar{\Delta}(\underline{q}_u) \quad , \tag{7.14}$$

where

$$\bar{\Delta}(\underline{q}_u) \equiv \int_0^\infty da_u \int_{-\infty}^\infty dp_u \; 2\pi m^2 \omega^2 a_u J_o(q_u a_u)\, \hat{\Delta}'(u) \,\Big|_{\nu_u = 0} \quad , \tag{7.15}$$

and we have used the relation

$$\sum_n J_{n+\nu} J_n = \delta^{Kr}(\nu).$$

Furthermore, eq. (7.13) yields

$$\bar{\Delta}(\underline{q}_h) = \sum_{n_h} \int_0^\infty da_h \int_{-\infty}^\infty dp_h \; 2\pi m^2 \omega^2 a_h$$

$$J_{n_h}(q_h a_h) \exp(i n_h \Psi_h)\, \hat{\Delta}(h). \tag{7.16}$$

Therefore eq. (7.14) may be reduced by means of this barring operation to

$$\bar{\Delta}(\underline{q}_h) = \Xi(\underline{q}_h) + \frac{4\pi e^2 N}{i\pi m\omega\Lambda} \sum_{n_h}$$

$$\cdot \int_0^\infty da_h \int_{-\infty}^{\infty} dp_h \; 2\pi m^2 \omega^2 a_h J_{n_h}(q_h a_h)$$

$$\cdot \exp(i n_h \psi_h)\, \delta_-(G_x - [(kp_h/m) + n_h\omega])$$

$$\cdot \int_0^\infty \frac{q_u dq_u}{q_u^2 + k^2 + \varkappa^2} \int_0^{2\pi} d\psi_u J_{n_h}(q_u a_h)$$

$$\cdot \exp(-i n_h \psi_u) \left[m\omega k \frac{\partial}{\partial p_h} + i(\underline{q}_u \times \underline{q}_h)^z \right.$$

$$\left. + \frac{n_h}{a_h} \frac{\partial}{\partial a_h} \right] \hat{\varphi}(a_h, p_h, \underline{k}_h = \underline{q}_u - \underline{q}_h)\, \bar{\Delta}(\underline{q}_u). \qquad (7.17)$$

This equation can be written in a notation that may be more familiar:

$$\bar{\Delta}(\underline{q}) = \Xi(q) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\underline{q}'\, K(\underline{q}, \underline{q}')\, \bar{\Delta}(\underline{q}'), \qquad (7.18)$$

where

$$K(\underline{q}, \underline{q}') = \frac{4\pi e^2 N}{i\Lambda(q'^2 + k^2 + \varkappa^2)} \int_0^\infty 2\pi p_\perp dp_\perp \int_{-\infty}^{\infty} dp_\parallel$$

$$\cdot \sum_{n=-\infty}^{\infty} \delta_-(G - [kp_\parallel/m + n\omega])$$

$$\cdot J_n\left(\frac{qp_\perp}{m\omega}\right) J_n\left(\frac{q' p_\perp}{m\omega}\right) \exp(in[\psi - \psi']$$

$$\cdot \left[k \frac{\partial}{\partial p_{\|}} + \frac{n m \omega}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} + \frac{i}{m\omega} (\underline{q}' \times \underline{q})_{\|} \right]$$

$$\cdot \hat{\varphi}(\underline{p}, \underline{k} = \underline{q}' - \underline{q}), \qquad (7.19)$$

and $\underline{p}$ is the kinetic momentum.

Equation (7.17) or (7.18) is of the Fredholm type, but the kernel K is rather complicated. We give it some study in the next section, but no detailed examination has yet been carried out.

SECTION 8

COMPARISON OF RESULTS WITH EXISTING KINETIC EQUATIONS

Our results reduce to the homogeneous kinetic equation(I. 11. 17), or (I. 11. 1), (II. 4. 9), when $\varphi(j)$ is independent of $\underline{Q}_j$. The easiest way to show this is to return to eq. (7. 1), and perform the integrations over the $\{ \underline{Q}_j \}$. We obtain factors $\delta(\underline{\ell}_h - \underline{\ell}_u)$, $\delta(\underline{\ell}_u - \underline{\ell})$, etc., so that the integrations over all but one of the $\{ \underline{\ell}_j \}$ are trivial; we may set $\ell_h = \ell_u = \ldots = \ell$ and $\beta_h = \beta_u = \ldots = \beta$. Then the Résibois summation procedure yields geometric series, and the final result is precisely eq. (II. 4. 9).

In order to see that this is plausible without going through the details, one should note that eq. (7. 17) reduces to

$$\bar{\Delta}(\underline{q}_h) = \Xi(\underline{q}_h)/\eta(x; \underline{k}_\perp = \underline{q}_h)$$

when $\hat{\varphi}(h) = \delta(\underline{k}_h)\varphi(h)$, if the normalization of φ is changed by a factor Σ. Roughly speaking, the term $q(\alpha)/\eta(\alpha)$ of eq. (II. 4. 9) comes from the term $\tilde{\Delta}(\alpha, h)$ of eq. (7. 8), and the integrand $\delta_-(G_{\alpha u})[q_{(1)}(u)/\eta(u)]/\eta^*(u)$ in the other term comes from the factor $\Gamma(u, h; x)$ in eq. (7. 8).

Let us now return to the diffusion problem. So far as we are aware, no equivalent results have yet been published. The only comparable results are those of Eleonskiǐ, Zyryanov, and Silin[12], especially their equation (17). They made a quantum mechanical calculation of the collision integral, using the first Born approximation and the Landau representation (Landau levels), and later took the classical limit. The collective effects were taken into account through the ad hoc introduction of a dielectric tensor. They simplified the results by assuming that $\varphi(j)$ is independent of Q_j^x, depending only on Q_j^y (or y_j^g, in our previous notation), a_j, p_j, and t. If we set the collective factor equal to 1 (so that the results are analogous to the Landau equation), their collision integral for the single species plasma may be written as follows:

$$\int dk \int dQ_h^y \int_0^\infty da_h \int_{-\infty}^{\infty} dp_h \, 2\pi m^2 \omega^2 a_h \sum_{n_\alpha} \sum_{n_h}$$

$$\cdot\left(\frac{e^4}{\pi^2}\right)\int_{\infty}^{\infty} d\ell^x \left(k\frac{\partial}{\partial p_\alpha} - \frac{\ell^x}{m\omega}\frac{\partial}{\partial Q^y_\alpha} + \frac{n_\alpha}{m\omega a_\alpha}\frac{\partial}{\partial a_\alpha}\right)$$

$$\cdot\left[\mathrm{Re}\ \delta_-\left(\frac{kp_\alpha}{m} + n_\alpha\omega - \frac{kp_h}{m} - n_h\omega\right)\right]$$

$$\cdot\left|\int d\ell^y_\alpha \int d\ell^y_h (k^2 + \ell^{x2} + \ell^y_\alpha \ell^y_h)^{-1}\right.$$

$$\cdot \exp(i[n_\alpha\beta_\alpha - n_h\beta_h])\exp(i[\ell^y_\alpha Q^y_\alpha - \ell^y_h Q^y_h])$$

$$\cdot J_{n_\alpha}(\ell_\alpha a_\alpha) J_{n_h}(\ell_h a_h)\Big|^2 \left(k\frac{\partial}{\partial p_\alpha} - k\frac{\partial}{\partial p_h} - \frac{\ell^x}{m\omega}\frac{\partial}{\partial Q^y_{\alpha h}}\right.$$

$$\left. + \frac{n_\alpha}{m\omega a_\alpha}\frac{\partial}{\partial a_\alpha} - \frac{n_h}{m\omega a_h}\frac{\partial}{\partial a_h}\right)\left[\frac{N}{\Lambda}\varphi(h)\right]\varphi(\alpha);$$

$$\ell^x = \ell^x_\alpha = \ell^x_h . \tag{8.1}$$

This is to be compared with our result (for $\varkappa \to 0$);

$$\mathrm{Re}\int dk \int dQ^y_h \int_0^\infty da_h \int_{-\infty}^\infty dp_h\, 2\pi m^2\omega^2 a_h \sum_{n_\alpha}\sum_{n_h}$$

$$\cdot\left(\frac{e^4}{\pi^2}\right)\int_{-\infty}^{\infty} d\ell^x\left(k\frac{\partial}{\partial p_\alpha} - \frac{\ell^x}{m\omega}\frac{\partial}{\partial Q^y_\alpha} + \frac{n_\alpha}{m\omega a_\alpha}\frac{\partial}{\partial a_\alpha}\right)$$

$$\cdot\delta_-\left(\frac{kp_\alpha}{m} + n_\alpha\omega - \frac{kp_h}{m} - n_h\omega\right)\left[\int d\ell^y_\alpha \int d\ell^y_h\right.$$

$$\cdot(k^2 + \ell^{x2} + \ell^{y2}_\alpha)^{-1}(k^2 + \ell^{x2} + \ell^{y2}_h)^{-1}$$

$$\cdot \exp(i[n_\alpha - n_h][\beta_\alpha - \beta_h])\exp(i[\ell^y_\alpha - \ell^y_h]\, Q^y_{\alpha h})$$

$$\cdot J_{n_\alpha}(\ell_\alpha a_\alpha) J_{n_\alpha}(\ell_h a_\alpha) J_{n_h}(\ell_\alpha a_h) J_{n_h}(\ell_h a_h)]$$

$$\cdot \left(k \frac{\partial}{\partial p_a} - k \frac{\partial}{\partial p_h} - \frac{\ell^x}{m\omega} \frac{\partial}{\partial Q^y_{\alpha h}} + \frac{n_\alpha}{m\omega a_\alpha} \frac{\partial}{\partial a_\alpha}\right.$$

$$\left. - \frac{n_h}{m\omega a_h} \frac{\partial}{\partial a_h}\right) \left[\frac{N}{\Lambda} \varphi(h)\right]\varphi(\alpha). \qquad (8.2)$$

Expressions (8.1) and (8.2) are very similar, but seem to be slightly different. We have not yet traced the origin of the differences.

In (8.1) or (8.2), the coefficient of $\partial^2/\partial {Q^y_\alpha}^2$ is proportional to ω^{-2}, or B^{-2}. Hence the results may be in qualitative agreement with Fick's law with a collisional diffusion coefficient. (See References (1), (2), and (12) for more complete discussions.) Our hope is that by properly taking into account the collective effects, as we have done, we may eventually be able to understand better the phenomenon of anomalous diffusion, where the diffusion coefficient decreases less rapidly than B^{-2} for increasing magnetic field strength.

SECTION 9

CONCLUSION

We have provided here a basis for kinetic theoretical studies of the diffusion of a plasma across a magnetic field. The main result contained in this report is a demonstration that the problem of deriving a kinetic equation for the one particle distribution function can be reduced to the problem of solving an equation of the Fredholm type (7.18). The demonstration, which is believed to be original, contains no explicit restriction on the amplitude or length scale of the inhomogeneity across the magnetic field.

Much further work, however, remains to be done on the refinement of the physical model, and on the qualitative and quantitative examination of the properties of the resulting equations.

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*Errata:

Page 1977, line 26: the last two lines of the expression for δL_{hu} should read as follows:

$$\{[m_\perp \ell^z \partial/\partial p_h^z - (\underline{\ell}^\perp \times \partial/\partial \underline{x}_h^{g\perp})^z$$

$$+ \ell^\perp (\cos[\theta_h - \beta]\, \partial/\partial a_h$$

$$-[1/a_h]\sin[\theta_h - \beta]\,\partial/\partial\theta_h] - [h \longleftrightarrow u]\}.$$

Page 2000, line 6 and line 9: "(8.15)" should be "(8.14)".

Page 2008, line 12: "q = 1, 2, 2, 2, 1" should be "q = 1, 2, 2, 1, 1". See also the footnote on page 130 of paper II.

† Errata:

Page 123, line 5: "m^2" should be "m^6".

line 10: "$(2\pi)^{-1}$" should be "$(2\pi)^{-2}$".

Page 125, Fig. 2: "$\prod_{\sigma}$" should be "$\prod_{\sigma=1}^{s-1}$".

Page 128, line 6: "m^2" should be "m^6".

Page 128, last paragraph: "first order e^2" should read "first order in e^2".

Page 129, line 21: "$\mathcal{J}_h\, \delta(G_{\alpha h}\varphi^{eq}(\alpha)\varphi^{eq}(h)$" should be "$\mathcal{J}_h\, \delta(G_{\alpha h})G_{\alpha h}\varphi^{eq}(\alpha)\varphi^{eq}(h)$".

Page 129, line 26: "$\left(\frac{\varkappa_d^2}{k^2+\varkappa^2}\right)^{s-1}$" should be "$\left(\frac{-\varkappa_d^2}{k^2+\varkappa^2}\right)^{s-1}$".

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13. ABSTRACT

The kinetic theory of Prigogine and Balescu, which was previously developed for a homogeneous plasma in a magnetic field, is now applied to the problem of diffusion of a plasma column across a magnetic field. The problem of deriving a kinetic equation is reduced to that of solving a two dimensional Fredholm-type integral equation with a complicated kernel. No restriction is made concerning the amplitude or length scale of the inhomogeneity across the magnetic field. The results are preliminary in nature, and suggestions are made for further refinements. Some comparisons with previously existing results are given.

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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SUPPLEMENTARY

INFORMATION

Research Report No. PIBMRI-1289-65 by Michael J. Haggerty
(Department of Physics, Polytechnic Institute of Brooklyn,
Contract No. NONR 839(38), ARPA Order No. 529,
August 10, 1965.)

On the Kinetic Theory of Diffusion of a
Plasma Column across a Magnetic Field.

<u>ERRATA</u> (as of 12 January 1966):

Page 2, fourth line after eq. (2.1): replace "effect" by "affect".

Page 2, third paragraph: replace "assumption (I.3.26)" by "assumptions (I.3.26) and (I.8.12)".

Page 2, end of third paragraph: replace "with W set equal to zero" by "with W chosen so as to cancel the contribution of $V(k^z=0)$ to the interaction matrix elements."

Page 2, second last line before footnote: replace "may be a less trivial problem" by "is a less trivial problem".

Page 7, add at bottom of page:

"It should be observed that the matrix elements that are diagonal with respect to the k's are not necessarily zero. However, in accordance with the discussion in Section 2, we ignore these contributions by replacing $(\ell^2 + [k_h - k_h']^2 + \kappa^2)^{-1}$ by 0 when $k_h = k_h'$."

Page 8, third line of eq. (4.1): "$\{z_j$" should read "$\{z_j\}$".

Page 9, left side of eq. (4.5) should read "$f_{r,s}(0)$".

Page 11, third line from bottom: replace $\sum_k$ by $\sum_{k(\neq 0)}$, and include a marginal note referring to the paragraph added at the bottom of page 7.

Page 12, line 3: "$k_u = k$" should read "$k_u = -k$".

Page 14, second line of eq. (5.6): replace $\sum_k$ by $\sum_{k(\neq 0)}$.

Page 17, add a dagger after "binary correlation function", and write the following footnc... at the bottom of the page:

"†The appearance of U_2 at this stage, instead of $f_{2,2}$, implies that the self-consistent electric field is being ignored (*cf.* Sections 2 and 3)."

Page 21, line 2: "G_s" should read "G_S".

Page 22, eq. (7.8): replace "dG_x" by "$(dG_x/2\pi)$".

Page 22, eq. (7.10): replace "$\phi(h)\ \Delta(\alpha,h;x)$" by "$\phi(h)\ \Delta(\alpha,u;x)$".

Page 25, eq. (7.18): "$\overline{\Xi}(\phi)$" should read "$\overline{\Xi}(\boldsymbol{\phi})$".

Page 25, add ") " at end of last line.

Page 26, last line of eq. (7.19): replace "$\underline{k}$" by "$\underline{k}_{\perp}$".

Page 28, eq. (8.1): replace $\left|\int d\ell_\alpha^y \,.....\, J_{n_h}(\ell_h a_h)\right|^2$ by $\left|\int d\ell^y (k^2+\ell^2)^{-1}\right.$

$\left.\exp(i[n_\alpha - n_h]\beta)\exp(i\ell^y Q_{\alpha h}^y)\, J_{n_\alpha}(\ell a_\alpha)\, J_{n_h}(\ell a_h)\right|^2$.

Page 29, line 3: "$k\partial/\partial p_a$" should be "$k\partial/\partial p_\alpha$".

Page 29, lines 5 and 6: replace "Expressions (8.1) and (8.2) ... differences." by "The results (8.1) and (8.2) are identical."

Page 31, add at bottom of page:

"Page 1996, eq. (7.21): replace $Z(e^2)$ by $Z(0)$, and add '$+ O(e^4)$'.

Eq. (7.22): add '$+ O(e^4)$'.

Lines 15 to 17: replace 'and, as it happens, is independent of e^2' by 'to first order'."

Page 32, last correction: replace "d" by "D" in two places.

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